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A function $u(x, y)$ is said to be harmonic throughout a certain region T of the x, y-plane if 1) it is analytic there, and 2) it satisfies Laplace's equation. It is well known that the first of these requirements may be replaced by the apparently much less restrictive one that the function u be continuous and have continuous first and second partial derivatives. In fact, it is sufficient to demand even less than the continuity of the second partial derivatives; but some demand beyond the mere existence of these derivatives must be made in order that it should be possible to apply Green's theorem.

It is my object in the present note to show how the theory of harmonic functions in two dimensions may be established without demanding even the existence of the second partial derivatives. This makes it necessary for us to take as our point of departure not Laplace's equation, since this involves the second partial derivatives whose existence we do not wish to demand, but some other characteristic property of harmonic functions. We select for this purpose the fundamental property that the integral of the normal derivative of a harmonic function, extended around a closed curve, is zero, and we thus have as our starting point not a differential equation of the second order, but a differentio-integral equation which involves only first partial derivatives.[1] It is needless to insist on the fact that it is this differentio-integral equation and not Laplace's equation which forms the starting point in most physical applications of harmonic functions.

We will lay down the following

Definition: u (x, y) *is said to be harmonic throughout a region* T *of the* x, y*-plane if it is single valued and continuous there, has continuous first partial derivatives, and, for any circle which lies wholly within* T,[2] *satisfies the equation:*

[1] We note in passing that a pure integral equation for harmonic functions is given by Poisson's Integral.

[2] That is, all points within the circle or on its circumference are to lie within the region T and not on its boundary.

$$(1) \qquad \int \frac{\partial u}{\partial n} ds = 0$$

the integral being extended around the circle, n *denoting the exterior normal and* s *the arc.*

It is clear that this includes all functions which are harmonic according to the ordinary definition. It is not so clear that it does not also include other functions; for since we have not demanded even the existence of the second derivatives of u, we cannot pass from (1) to Laplace's equation by the familiar application of Green's theorem.

THEOREM I. *On the circumferences of any two concentric circles lying wholly in* T *the average values of* u *are the same.*[3]

Call the radii of the circles a and b $(a < b)$ and use polar coördinates (r, ϕ) with pole at the centre of the circles. Then by our definition

$$\int_0^{2\pi} \frac{\partial u}{\partial r} d\phi = 0 \qquad (a \leqq r \leqq b)$$

$$\therefore \qquad \int_a^b \left[\int_0^{2\pi} \frac{\partial u}{\partial r} d\phi \right] dr = 0.$$

Since $\partial u / \partial r$ is a continuous function of (r, ϕ) we may reverse the order of integration. Denoting by u_a and u_b respectively the values of u on the smaller and larger circle, we thus get:

$$\int_0^{2\pi} [u_b - u_a] d\phi = 0.$$

from which our theorem follows at once.

By allowing a to approach zero as its limit we get

THEOREM II. *The average value of* u *on the circumference of any circle lying wholly in* T *is the value of* u *at the centre of this circle.*

This theorem we next state in the following generalized form:

THEOREM III. *If* u *is continuous within and on the circumference of a circle* C, *and is harmonic within it, then the average value of* u *on the circumference of* C *is its value at the centre of* C.

For the average value of u on the circumference of a circle a little smaller than C and concentric with it will, on account of the continuity

[3] The proofs here given of this theorem and the next were used by the writer in a paper on Gauss's Third Proof of the Fundamental Theorem of Algebra. Bull. Amer. Math. Soc., May, 1895.

of u, approach the average value of u on the circumference of C as the smaller circle approaches C as its limit. From this fact and Theorem II our theorem follows.

THEOREM IV. *If* u *is harmonic within* T, *and the region* T *is carried over by inversion into a region* T′; *and if we define a new function* u′ *in* T′ *so that it has the same value at every point of* T′ *that* u *has at the corresponding point of* T *then* u′ *is harmonic in* T′.

In the first place, inversion being an analytic transformation, u' is evidently continuous throughout T' and has continuous first partial derivatives there. Now consider any circle C' lying wholly in T', and denote by s' the arc and by n' the exterior normal of this circle. If C is the circle in T of which C' is the inverse and we consider corresponding points on these two circles, we have, since inversion is a conformal transformation, after neglecting infinitesimals of higher order:

$$\frac{ds'}{dn'} = \frac{ds}{dn}.$$

Accordingly

$$\int \frac{\partial u'}{\partial n'}\, ds' = \int \frac{\partial u}{\partial n}\, ds = 0,$$

the first integral being taken around C', the second around C. This completes the proof that u' is harmonic.

THEOREM V. *If* u *is continuous within and on the circumference of a circle* C, *and is harmonic within* C, *then the value of* u *at any point* P *within* C *is given by the integral:*

$$\frac{1}{2\pi}\int_{\theta=0}^{\theta=2\pi} u_\theta\, d\theta, \tag{2}$$

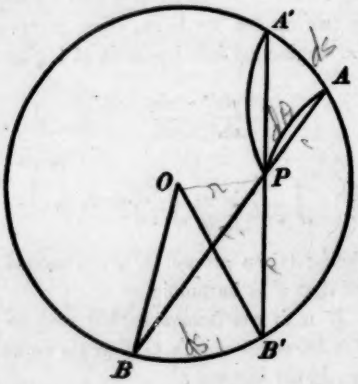
where θ *denotes the angle between a fixed and a variable circular arc which both start from* P *and which both cut* C *at right angles, and* u_θ *denotes the value of* u *at the point where this variable arc cuts* C.

In order to prove this, let us construct the point Q inverse to P with regard to C. If we invert the whole figure with regard to a circle having Q as centre, P goes over into a point P', C into a circle C' having P' as centre, and the fixed and variable arcs referred to in the theorem go over into fixed and variable radii of C' making the variable angle θ with one another. Finally, we define a function u' within and on the circumference of C' so that it has the same value at every point that u has at the corresponding point of C. This function u' will then be continuous

within and on the circumference of C', and by theorem IV is harmonic within C'. Accordingly its average on the circumference of C', which is obviously given by (2), is equal by III to its value at P', that is to the value of u at P. Thus our theorem is proved.

Formula (2) is merely an unfamiliar form of Poisson's Integral.[4] This may readily be shown by means of the following lemma, whose extremely simple proof we suppress:[5]

LEMMA; *If from a point* P *within a circle* C *two arcs are drawn orthogonal to* C *and meeting it at* A *and* A', *the angle between these arcs is measured by the arc* BB' *of* C *intercepted between the straight lines* AP *and* A'P *produced.*



Applying this to the case in which AA' is the element of arc ds, the angle between the two arcs PA and PA' will be $d\theta$. Denoting the arc BB' by ds_1, and the radius of the circle by R, this lemma tells us that $ds_1 = Rd\theta$. Accordingly (2) reduces to

$$(3) \qquad \frac{1}{2\pi R}\int u_\rho\, ds_1,$$

the integral being extended around the circle and u_ρ denoting the same thing as u_ρ. This is the form of Poisson's Integral given by Schwarz on p. 360, Vol. 2, of his collected works.

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[4] See a *Note on Poisson's Integral*, by the writer, Bull. Amer. Math. Soc., June, 1898, where a somewhat different treatment is given.

[5] See a paper by the writer on Fourier's Series in the Annals of Mathematics for January, 1906.

This form may be still further changed by means of the proportion $ds_1/\rho_1 = ds/\rho$, where $\rho_1 = PB$, $\rho = PA$. It may thus be written:

$$\frac{1}{2\pi R}\int u_s \frac{\rho_1 \rho}{\rho^2}\, ds.$$

Now, $\rho_1\rho = (R - r)(R + r)$, where r denotes the distance from P to the centre of C. Thus, finally, (2) takes the form:

$$\frac{1}{2\pi R}\int u_s \frac{R^2 - r^2}{\rho^2}\, ds, \tag{4}$$

which is Poisson's original form.

Since $(R^2 - r^2)/\rho^2$ is an analytic function of the coördinates (x, y) of P, the same will be true of the whole integral (4), and therefore $u(x, y)$ is analytic throughout the interior of the circle C. Moreover, $(R^2 - r^2)/\rho^2$ may be seen, by direct differentiation or otherwise, to satisfy Laplace's equation. Hence we infer that the same is true of the function $u(x, y)$ represented by (4).

We have been assuming that u is harmonic merely within C. If, however, it is harmonic throughout any region T, we can now infer at once that it is analytic at any point P of this region, and satisfies Laplace's equation there by merely surrounding P by a small circle and applying the results just established to this circle. Thus we have proved

THEOREM VI. *If* u (x, y) *is harmonic within any region* T, *it is analytic and satisfies Laplace's equation at every point within* T.

We have thus established the identity of the class of functions defined above as harmonic with the class ordinarily so defined.

In conclusion I add the following remarks on the result thus obtained.

1) In the definition of a harmonic function it is obviously not necessary to demand that (1) hold for *all* circles contained in T. All that is essential in the reasoning we have given will apply at once if we start from the following definition:

The function u (x, y) *is said to be harmonic at a point* (x_1, y_1), *if it is possible to surround this point by a region so small that* u *is continuous and has continuous first partial derivatives there, and for every circle contained in this region satisfies equation* (1). *The function* u *is said to be harmonic throughout a region* T *if it is harmonic at every point of this region.*

2) From the point of view of the physicist it may be desirable to demand in the definition *more* than was demanded above, namely, that (1) hold for all regular [6] closed curves lying within T, not merely for all circles. Since it is well known, and readily proved by an application of Green's theorem, that (1) is satisfied for all such curves when u is any harmonic function, it is clear that this will not affect the class of functions covered by the definition. Whether included as part of the definition or proved as a theorem, the fact just stated enables us to see almost intuitively that a harmonic function remains harmonic when the plane is subjected to any conformal transformation, the reasoning being the same as was used in the proof of theorem IV above.

3) The introduction of this general idea of conformal transformation enables us to extend at once our definition of harmonic functions to the case in which T is a region not in the plane but on any analytic curved surface.[7] The definition of a harmonic function on such a curved region T would be identical with the one given above for a plane region, if we understand the partial derivatives there referred to to mean the derivatives along the surface, and if we require that (1) hold for all regular (or even analytic) closed curves on T. If now we transform the region T, or any part of it, conformally onto the plane, we see at once that the function thus defined in the plane is harmonic, and hence the function on the curved surface was analytic there. — Here, as before, we may if we choose require that (1) be fulfilled only for certain classes of closed curves. There seems, in general, no object in doing this. In the case of the sphere, however, the developments given above can be carried through almost word for word if we restrict our attention to circles.

4) Finally, it may be mentioned that this method of approaching the subject is particularly convenient in connection with the theory of conjugate functions. Two functions u and v are said to be conjugate in a plane region T if throughout this region they are single valued, continuous, and have continuous first partial derivatives and satisfy the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

If we knew that u and v had continuous second partial derivatives, we could infer at once that they are both harmonic. To establish the con-

[6] A curve is said to be regular if it is continuous, and is made up of a finite number of arcs each of which has a continuously turning tangent at every point.

[7] Cf. C. Neumann, Math. Ann., **10**, 569 (1876).

tinuity of these derivatives, recourse is ordinarily had to the theory of functions of a complex variable. Instead of this, we may first deduce from the last written differential equations the relation

$$\frac{\partial u}{\partial \sigma} = \frac{\partial v}{\partial \tau}$$

where σ is any direction and τ the direction which makes a positive right angle with σ. From this we infer that

$$\int \frac{\partial u}{\partial n} ds = \int \frac{\partial v}{\partial s} ds, \qquad \int \frac{\partial v}{\partial n} ds = -\int \frac{\partial u}{\partial s} ds.$$

These integrals will therefore vanish when extended around a closed curve, since u and v are both single valued. From this it follows that u and v are both harmonic.

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